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XXIII. *On the Conditions for the existence of Three Equal Roots, or of Two Pairs of Equal Roots, of a Binary Quartic or Quintic.* By A. CAYLEY, F.R.S.

Received November 26, 1867,—Read January 9, 1868.

IN considering the conditions for the existence of given systems of equalities between the roots of an equation, we obtain some very interesting examples of the composition of relations. A relation is either onefold, expressed by a single equation $U=0$, or it is, say k -fold, expressed by a system of k or more equations. Of course, as regards onefold relations, the theory of the composition is well known: the relation $UV=0$ is a relation compounded of the relations $U=0$, $V=0$; that is, it is a relation satisfied if, and not satisfied unless one or the other of the two component relations is satisfied. The like notion of composition applies to relations in general; viz., the compound relation is a relation satisfied if, and not satisfied unless one or the other of the two component relations is satisfied. I purposely refrain at present from any further discussion of the theory of composition. I say that the conditions for the existence of given systems of equalities between the roots of an equation furnish instances of such composition; in fact, if we express that the function $(*\chi x, y)^n$, and its first-derived function in regard to x , or, what is the same thing, that the first-derived functions in regard to x , y respectively, have a common quadric factor, we obtain between the coefficients a certain twofold relation, which implies either that the equation $(*\chi x, y)^n=0$ has three equal roots, or else that it has two pairs of equal roots; that is, the relation in question is satisfied if, and it is not satisfied unless there is satisfied either the relation for the existence of three equal roots, or else the relation for the existence of two pairs of equal roots; or the relation for the existence of the quadric factor is compounded of the last-mentioned two relations. The relation for the quadric factor, for any value whatever of n , is at once seen to be expressible by means of an oblong matrix, giving rise to a series of determinants which are each to be put $=0$; the relation for three equal roots and that for two pairs of equal roots, in the particular cases $n=4$ and $n=5$, are given in my “Memoir on the Conditions for the existence of given Systems of Equalities between the roots of an Equation,” Phil. Trans. vol. cxlvii. (1857), pp. 727–731; and I propose in the present Memoir to exhibit, for the cases in question $n=4$ and $n=5$, the connexion between the compound relation for the quadric factor with the component relations for the three equal roots and for the two pairs of equal roots respectively.

Article Nos. 1 to 8, the Quartic.

1. For the quartic function

$$(a, b, c, d, e\chi x, y)^4,$$

4 K 2

the condition for three equal roots, or, say, for a root system 31, is that the quadriinvariant and the cubinvariant each of them vanish, viz. we must have

$$I = ae - 4bd + 3c^2 = 0,$$

$$J = ace - ad^2 - b^2e + 2bcd - c^3 = 0.$$

2. The condition for two pairs of equal roots, or for a root system 22, is that the cubicovariant vanishes identically, viz. representing this by

$$(A, B, 5C, 10D, 5E, F, G)(x, y)^6 = 0,$$

we must have

$$A = a^2d - 3abc + 2b^3 = 0,$$

$$B = a^2e + 2abd - 9ac^2 + 6b^2c = 0,$$

$$C = abe - 3acd + 2b^2d = 0,$$

$$D = -ad^2 + b^2e = 0,$$

$$E = -ade + 3bce - 2bd^2 = 0,$$

$$F = -ae^2 - 2bde + 9c^2e - 6cd^2 = 0,$$

$$G = -be^2 + 3cde - 2d^3 = 0.$$

3. But the condition for the common quadric factor is

$$\begin{vmatrix} a, & 3b, & 3c, & d \\ b, & 3c, & 3d, & e \\ a, & 3b, & 3c, & d \\ b, & 3c, & 3d, & e \end{vmatrix} = 0,$$

and the determinants formed out of this matrix must therefore vanish for $(I, J) = 0$, and also for $(A, B, C, D, E, F, G) = 0$, that is, the determinants in question must be syzygetically related to the functions (I, J) , and also to the functions (A, B, C, D, E, F, G) .

4. The values of the determinants are—

1234 = 3 ×	1235 = 3 ×	1245 =	1345 = 3 ×	2345 = 3 ×
+ 1 a^2ce	- 1 a^2de	- 1 a^2e^2	- 1 abe^2	+ 1 ace^2
- 3 a^2d^2	+ 4 $abce$	+ 2 $abde$	+ 4 $acde$	- 1 ad^2e
- 1 ab^2e	+ 1 abd^2	+ 9 ac^2e	- 3 ad^3	- 3 b^2e^2
+ 14 $abcd$	- 3 ac^2d	- 9 acd^2	+ 1 b^2de	+ 14 $bcdde$
- 9 ac^3	- 3 b^3e	- 9 b^2ce	- 3 bc^2e	- 8 bd^3
- 8 b^3d	+ 2 b^2cd	+ 8 b^2d^2	+ 2 bcd^2	- 9 c^3e
+ 6 b^2c^2				+ 6 c^2d^2

5. The syzygetic relation with (I, J) is given by means of the identical equation

$$\begin{vmatrix} y^4, & -4xy^3, & 6x^2y^2, & -4x^3y, & x^4 \\ & a, & 3b, & 3c, & d \\ & b, & 3c, & 3d, & e \\ a, & 3b, & 3c, & d, & \\ b, & 3c, & 3d, & e, & \end{vmatrix} = -6I \cdot \tilde{H}U + 9J \cdot U,$$

or, as this may be written,

$$(1234, 1235, 1245, 1345, 2345)\chi(x, y)^4 = -6I.\tilde{H}U + 9J.U,$$

where $\tilde{H}U$ is the Hessian of U ,

$$= \left(\begin{array}{c|c|c|c|c} ac & 2ad & ae & 2be & ce \\ -b^2 & -2bc & +2bd & -2cd & -d^2 \end{array} \right) \chi(x, y)^4.$$

6. That is, we have

$$\begin{aligned} 1234 &= (ac - b^2, \quad a\chi - 6I, 9J), \\ 4. 1235 &= (2ad - 2bc, \quad 4b\chi - 6I, 9J), \\ 6. 1245 &= (ae + 2bd - 3c^2, \quad 6c\chi - 6I, 9J), \\ 4. 1345 &= (2be - 2cd, \quad 4d\chi - 6I, 9J), \\ 2345 &= (ce - d^2, \quad e\chi - 6I, 9J). \end{aligned}$$

7. The determinants thus vanish if $(I, J)=0$, that is, for the root system 31; they will also vanish without this being so, if only

$$\left(\frac{3J}{2I}=\right) \frac{ac-b^2}{a} = \frac{ad-bc}{2b} = \frac{ae+2bd-3c^2}{6c} = \frac{be-cd}{2d} = \frac{ce-d^2}{e};$$

and we may omit the first member $\left(\frac{3J}{2I}=\right)$, since if the remaining terms are equal to each other they will also be $=\frac{3J}{2I}$. The equations may then be written

$$\left\| \begin{array}{ccccc} ac-b^2, & ad-bc, & ae+2bd-3c^2, & be-cd, & ce-d^2 \\ a, & 2b, & 6c, & 2d, & e \end{array} \right\| = 0,$$

and the ten equations of this system reduce themselves (as it is very easy to show) to the seven equations

$$(A, B, C, D, E, F, G)=0,$$

which, as above mentioned, are the conditions for the root system 22.

8. It may be added that we have

	A	B	C	D	E	F	G
$\frac{1}{3}.1234=$		c	$-4b$	$+3a$			
$\frac{1}{3}.1235=$			c	$-3b$	$+a$		
$0=$		d	$-3c$		$+a$		
$1245=$		$-e$	$+4d$	$-3c$			
$0=$		$-e$		$+6c$		$-a$	
$0=$			$-d$	$+3c$	$-b$		
$\frac{1}{3}.1345=$			$-e$	$+3d$	$-c$		
$0=$			$-e$		$+3c$	$-b$	
$\frac{1}{3}.2345=$				$-3e$	$+4d$	$-c$	

where it is to be noticed that the four equations having the left-hand side $=0$, give

B:C:D:E:F proportional to the determinants of the matrix

$$\begin{vmatrix} d, & -3c, & , & a \\ -e, & ., & 6c, & ., & -a \\ & -d, & 3c, & -b, \\ & -e, & ., & +3c, & -b \end{vmatrix};$$

the determinants in question contain each the factor c , and omitting this factor, the system shows that B, C, D, E, F are proportional to their before-mentioned actual values.

Article Nos. 9 to 15, the Quintic.

9. For the quintic function

$$(a, b, c, d, e, f \mathfrak{X}x, y)^5,$$

the condition of a root system 41 is that the covariant, Table No. 14, shall vanish, viz. we must have

$$A=2(ae-4bd+3c^2)=0,$$

$$B= af-3be+2cd=0,$$

$$C=2(bf-4ce+3d^2)=0.$$

10. The condition of a root system 32 is that the following covariant, viz.

$$3(\text{No. 13})^2(\text{No. 14})-25(\text{No. 15})^2,$$

shall vanish, where

No. 13 = $(a, b, c, d, e, f \mathfrak{X}x, y)^5$, the quintic itself.

$$\text{No. 14} = \left(\begin{array}{c|c|c} ae & af & bf \\ -4\ bd & -3\ be & -4\ ce \\ +3\ c^2 & +2\ cd & +3\ d^2 \end{array} \right) \mathfrak{X}x, y)^2.$$

$$\text{No. 15} = \left(\begin{array}{c|c|c|c|c|c|c} ac & 3\ ad & 3\ ae & af & 3\ bf & 3\ cf & df \\ -b^2 & -3\ bc & +3\ bd & +7\ be & +3\ ce & -3\ de & -e^2 \end{array} \right) \mathfrak{X}x, y)^6.$$

11. The developed expression of the foregoing function is as follows:—

2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120	121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139	140	141	142	143	144	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	160	161	162	163	164	165	166	167	168	169	170	171	172	173	174	175	176	177	178	179	180	181	182	183	184	185	186	187	188	189	190	191	192	193	194	195	196	197	198	199	200	201	202	203	204	205	206	207	208	209	210	211	212	213	214	215	216	217	218	219	220	221	222	223	224	225	226	227	228	229	230	231	232	233	234	235	236	237	238	239	240	241	242	243	244	245	246	247	248	249	250	251	252	253	254	255	256	257	258	259	260	261	262	263	264	265	266	267	268	269	270	271	272	273	274	275	276	277	278	279	280	281	282	283	284	285	286	287	288	289	290	291	292	293	294	295	296	297	298	299	300	301	302	303	304	305	306	307	308	309	310	311	312	313	314	315	316	317	318	319	320	321	322	323	324	325	326	327	328	329	330	331	332	333	334	335	336	337	338	339	340	341	342	343	344	345	346	347	348	349	350	351	352	353	354	355	356	357	358	359	360	361	362	363	364	365	366	367	368	369	370	371	372	373	374	375	376	377	378	379	380	381	382	383	384	385	386	387	388	389	390	391	392	393	394	395	396	397	398	399	400	401	402	403	404	405	406	407	408	409	410	411	412	413	414	415	416	417	418	419	420	421	422	423	424	425	426	427	428	429	430	431	432	433	434	435	436	437	438	439	440	441	442	443	444	445	446	447	448	449	450	451	452	453	454	455	456	457	458	459	460	461	462	463	464	465	466	467	468	469	470	471	472	473	474	475	476	477	478	479	480	481	482	483	484	485	486	487	488	489	490	491	492	493	494	495	496	497	498	499	500	501	502	503	504	505	506	507	508	509	510	511	512	513	514	515	516	517	518	519	520	521	522	523	524	525	526	527	528	529	530	531	532	533	534	535	536	537	538	539	540	541	542	543	544	545	546	547	548	549	550	551	552	553	554	555	556	557	558	559	560	561	562	563	564	565	566	567	568	569	570	571	572	573	574	575	576	577	578	579	580	581	582	583	584	585	586	587	588	589	590	591	592	593	594	595	596	597	598	599	600	601	602	603	604	605	606	607	608	609	610	611	612	613	614	615	616	617	618	619	620	621	622	623	624	625	626	627	628	629	630	631	632	633	634	635	636	637	638	639	640	641	642	643	644	645	646	647	648	649	650	651	652	653	654	655	656	657	658	659	660	661	662	663	664	665	666	667	668	669	670	671	672	673	674	675	676	677	678	679	680	681	682	683	684	685	686	687	688	689	690	691	692	693	694	695	696	697	698	699	700	701	702	703	704	705	706	707	708	709	710	711	712	713	714	715	716	717	718	719	720	721	722	723	724	725	726	727	728	729	730	731	732	733	734	735	736	737	738	739	740	741	742	743	744	745	746	747	748	749	750	751	752	753	754	755	756	757	758	759	760	761	762	763	764	765	766	767	768	769	770	771	772	773	774	775	776	777	778	779	780	781	782	783	784	785	786	787	788	789	790	791	792	793	794	795	796	797	798	799	800	801	802	803	804	805	806	807	808	809	810	811	812	813	814	815	816	817	818	819	820	821	822	823	824	825	826	827	828	829	830	831	832	833	834	835	836	837	838	839	840	841	842	843	844	845	846	847	848	849	850	851	852	853	854	855	856	857	858	859	860	861	862	863	864	865	866	867	868	869	870	871	872	873	874	875	876	877	878	879	880	881	882	883	884	885	886	887	888	889	890	891	892	893	894	895	896	897	898	899	900	901	902	903	904	905	906	907	908	909	910	911	912	913	914	915	916	917	918	919	920	921	922	923	924	925	926	927	928	929	930	931	932	933	934	935	936	937	938	939	940	941	942	943	944	945	946	947	948	949	950	951	952	953	954	955	956	957	958	959	960	961	962	963	964	965	966	967	968	969	970	971	972	973	974	975	976	977	978	979	980	981	982	983	984	985	986	987	988	989	990	991	992	993	994	995	996	997	998	999	1000
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12. The conditions for the common quadric factor are

$$\begin{vmatrix} a, & 4b, & 6c, & 4d, & e \\ a, & 4b, & 6c, & 4d, & e \\ b, & 4c, & 6d, & 4e, & f \\ b, & 4c, & 6d, & 4e, & f \end{vmatrix} = 0,$$

the several determinants whereof are given in Table No. 27 of my "Third Memoir on Quantics," Philosophical Transactions, vol. cxlvi. (1856), pp. 627–647.

13. These determinants must therefore vanish, for $(A, B, C)=0$, and also for $(A, B, \dots L, M)=0$, that is, they must be syzygetically connected with (A, B, C) , and also with $(A, B, \dots L, M)$. The relation to (A, B, C) is in fact given in the Table appended to Table No. 27, viz. this is

	C ×	+ B ×	+ A ×
1234=	+ 6 a^2	− 12 ab	+ 16 ac − 10 b^2
1235=	+ 6 ab	− 2 ac − 10 b^2	+ 6 ad
1236=	− 2 ac + 8 b^2	+ 6 ad − 18 bc	− 2 df + 8 e^2
1245=	+ 18 ac	− 6 ad − 30 bc	+ 8 ae + 10 bd
1246=	+ 12 bc	+ 4 ae − 4 bd − 24 c^2	+ 4 be + 8 cd
1345=	+ 24 ad	− 8 ae − 40 bd	+ 4 af + 20 be
1256=	− 1 ae + 4 bd + 3 c^2	+ 1 af + 5 be − 18 cd	− 1 bf + 4 ce + 3 d^2
2345=	+ 20 ae + 40 bd − 30 c^2	− 80 be + 20 cd	+ 20 bf + 40 ce − 30 d^2
1346=	+ 4 ae + 8 bd + 6 c^2	− 36 cd	+ 4 bf + 8 ce + 6 d^2
2346=	+ 4 af + 20 be	− 8 bf − 4 ce	+ 24 cf
1356=	+ 4 be + 8 cd	+ 4 bf − 4 ce − 24 d^2	+ 12 de
2356=	+ 8 bf + 10 ce	− 6 cf − 30 de	+ 18 df
1456=	+ 6 ce	+ 6 cf − 18 de	− 2 df + 8 e^2
2456=	+ 6 cf	− 2 df − 10 e^2	+ 6 ef
3456=	+ 16 df − 10 e^2	− 12 ef	+ 6 f^2

14. Between the expressions A, B , &c., and 1234, 1235, &c., there exist relations the form of which is indicated by the following Table:

[illegible]

viz. these relations are of the form

$$\begin{aligned} & ()c\mathbb{A}+()b\mathbb{B}+()a\mathbb{C} & + ()a.1234 & =0, \\ & ()d\mathbb{A}+()c\mathbb{B}+()b\mathbb{C}+()a\mathbb{D} & + ()b.1234+()a.1235=0, \\ & \&c. \end{aligned}$$

where the brackets $()$ denote numerical coefficients, determinate as to their ratios,

15. Assuming the existence of these relations, we have for the determination of the numerical coefficients in each relation a set of linear equations, which are shown by the following Tables, viz. referring to the Table headed $c\mathfrak{A}$, $b\mathfrak{B}$, $a\mathfrak{C}$, $a.1234$, if the multipliers of terms respectively be A, B, C, X, then the Table denotes the system of linear equations

$$0 \quad A \quad + 3 \quad B \quad + 33 \quad C \quad + 0 \quad X = 0,$$

$$3 \quad A \quad + 0 \quad B \quad - 102 \quad C \quad - 16 \quad X = 0,$$

&c.,

that is, nine equations to be satisfied by the ratios of the coefficients A, B, C, X, and which are in fact satisfied by the values at the foot of the Table, viz.

$$A : B : C : X = 66 : -11 : +1 : +6.$$

There would be in all fourteen Tables, but as those for the second seven would be at once deducible by symmetry from the first seven, I have only written down the seven Tables; the solutions for the first and second Tables were obtained without difficulty, but that for the third Table was so laborious to calculate, and contains such extraordinarily high numbers, that I did not proceed with the calculation, and it is accordingly only the first, second, and third Tables which have at the foot of them respectively the solutions of the linear equations.

16. The results given by these three Tables are, of course,

$$66c\mathfrak{A} - 11b\mathfrak{B} + 1a\mathfrak{C} + 6a.1234 = 0,$$

$$330d\mathfrak{A} + 110c\mathfrak{B} - 55b\mathfrak{C} + 9a\mathfrak{D} - 105 a.1235 = 0,$$

$$+ 266478575 \quad e\mathfrak{A}$$

$$- 617359490 \quad d\mathfrak{B}$$

$$+ 144200810 \quad c\mathfrak{C}$$

$$+ \quad 9656911 \quad b\mathfrak{D}$$

$$+ \quad 9090785 \quad a\mathfrak{E}$$

$$- 721004050 \quad c.1234$$

$$+ \quad 90914175 \quad b.1235$$

$$- 160758675 \quad a.1245$$

$$+ 11559295 \quad a.1236 = 0.$$

It is to be noticed that the nine coefficients of this last equation were obtained from, and that they actually satisfy, a system of fourteen linear equations; so that the correctness of the result is hereby verified.

17. The seven Tables are

First Table.

	cA	bB	aC	$a.1234$
a^3bf		+ 3	+ 33	
a^3ce	+ 3		-102	- 16
a^3d^2			-216	+ 36
a^2be^2		+ 21	+135	+ 16
a^2bcd	-12	-144	+120	-152
a^2c^3	-16		+480	+ 96
ab^3d		+ 30	-150	+ 80
ab^2c^2	+50	+240	-300	- 60
b^4c	-25	-150		
	+66	-11	+1	+ 6

Second Table.

	dA	cB	bC	aD	$a.1235$	$b.1234$
a^3cf		+ 3		+ 10	- 4	
a^3de	+ 3			- 390	+24	
a^3b^2f			+ 33	+ 155	+ 4	
a^2bce		+ 21	-102	+ 100	-84	- 16
a^2bd^2	-12		-216	- 600	-24	+ 36
a^2c^2d	-16	-144		+1600	+64	+ 16
ab^3e			+135	+ 125	+60	
ab^2cd	+50	+ 30	+120	-1000	-40	-152
abc^3		+240	+480			+ 96
b^4d	-25		-150			+ 80
b^3c^2		-150	-300			- 60
	+330	+110	-55	+9	-105	0

Third Table.

	eA	dB	cC	bD	aE	$c.1234$	$b.1235$	$a.1245$	$a.1236$
a^3df		+ 3			- 90			- 6	+ 6
a^3e^2	+ 3				- 195			+16	
a^2bcf			+ 33	+ 10	+ 360		- 4	+ 6	-22
a^2bde	-12	+ 21		- 390	-1500		+24	-26	- 6
a^2c^2e	-16		-102		+ 900	- 16		-96	+16
a^2cd^2		-144	-216		+1800	+ 36		+96	
ab^3f				+ 155	+ 225		+ 4		+16
ab^2ce	+50		+135	+ 100		+ 16	-84	+90	-10
ab^2d^2		+ 30		- 600	-1500		-24	-80	
abc^2d		+240	+120	+1600		-152	+64		
ac^4			+480			+ 96			
b^4e	-25			+ 125			+60		
b^3cd		-150	-150	-1000		+ 80	-40		
b^3c^2			-300			- 60			
	+266478575	-617359490	+144200810	+9656911	+9090785	-721004050	+90914175	-160758675	+11559295

Fourth Table.

	fA	eB	dC	cD	bE	aF	$d.1234$	$c.1235$	$b.1236$	$b.1245$	$a.1246$	$a.1345$
a^3ef	+ 3	+ 3				- 114					+ 4	
a^2bdf	-12		+ 33		- 90	- 264			+ 6	- 6	- 4	- 24
a^2be^2		+ 21			- 195	- 990				+16	- 4	+ 64
a^2c^2f	-16			+ 10		+ 468		- 4			-24	+ 24
a^2cde		-144	-102	- 390		+1320	- 16	+24			+24	-208
a^2d^3			-216			+1080	+ 36					+144
ab^2cf	+50			+ 155	+ 360	+ 900		+ 4	-22	+ 6	+24	
ab^2de		+ 30	+135		-1500	-2700	+ 16		- 6	-26	-20	- 40
abc^2e		+240		+ 100	+ 900	+ 900		-84	+16	-96		+ 60
$abcd^2$			+120	- 600	+1800	- 600	-152	-24		+96		- 40
ac^3d			+480	+1600			+ 96	+64				
b^4f	-24				+ 225				+16			
b^3ce		-150		+ 125				+60	-10	+90		
b^3d^2			-150		-1500		+ 80			-80		
b^2c^2d			-300	-1000			- 60	-40				

Fifth Table.

	fB	eC	dD	cE	bF	aG	$e.1234$	$d.1235$	$c.1236$	$c.1245$	$b.1246$	$b.1345$	$a.1256$	$a.2345$	$a.1346$
a^3f^2	+ 3					- 19							+ 1		
a^2bef	+ 21	+ 33			- 114	- 608					+ 4		- 2		+16
a^2cdf	-144		+ 10	- 90		+ 537		- 4	+ 6	- 6			-16	+ 20	-36
a^2ce^2		-102		- 195		- 245	- 16			+16			+16	- 80	-16
a^2d^3e		-216	- 390			+1740	+ 36	+24					+16	+ 60	+36
ab^2df	+ 30		+ 155		- 264	- 245		+ 4			- 4	- 24	-15	- 80	-16
ab^2e^2		+135			- 990	-1700	+ 16				- 4	+ 64		+240	
abc^2f	+240			+ 360	+ 468	+1740			-22	+ 6	-24	+ 24		+ 60	+36
$abcde$		+120	+ 100	-1500	+1320	-2000	-152	-84	- 6	-26	+24	-208		-860	-20
abd^3			- 600		+1080	+ 600		-24				+144		+960	
ac^3e		+480		+ 900		+ 600	+ 96		+16	-96				+960	
ac^2d^2			+1600	+1800		- 400		+64		+96				-320	
b^3cf	-150			+ 225	+ 900				+16		+24				
b^3de		-150	+ 125		-2700		+ 80	+60			-20	- 40			
b^2c^2e		-300			+ 900		- 60		-10	+90		+ 60			
b^2cd^2			-1000	-1500	- 600			-40		-80		- 40			

Sixth Table.

f	e	d	c	b	a	$f.1234$	$e.1235$	$d.1236$	$c.1245$	$b.1256$	$a.2346$	$b.1346$	$a.1356$
a^2bf^2	+ 33	+ 10	-	- 114	-	- 114	- 16	- 4	+	6	+	6	+
a^2cef	- 102	-	90	-	-	- 264	+ 36	+ 24	-	- 6	-	24	-
a^2d^2f	- 216	- 390	- 195	-	-	468	+ 16	+ 4	+	+ 16	+	24	+
a^2de^2	+ 135	+ 155	+ 360	-	-	990	- 152	- 84	-	6	+	64	+
ab^2cf	+ 120	+ 100	-	- 990	-	- 1320	- 152	- 84	-	6	+	208	+
abc^2f	+ 480	- 600	- 1500	+	+	- 2700	96	- 24	-	- 26	+	- 40	-
abd^2e		+ 1600	+ 900	+ 1320	+	+ 900	+	+ 64	+	16	+	60	+
ac^2f		+ 1800	+ 1800	+ 1080	-	+ 1080	+	+	+	24	+	144	+
acd^3	- 150	+	225	-	-	600	+	+	+	- 208	-	40	-
b^3df	- 300	- 1000	- 1500	+	+	-	80	+ 60	+	- 80	+	-	-
b^3e^2		+ 125	-	900	-	- 1700	60	- 40	+	240	+	36	+
b^2e^2f		+	225	+	+	+ 1740	-	- 40	-	60	+	20	-
b^2cde		- 1000	- 1500	- 2700	+	- 2000	+	+	+	960	+	960	+
b^2d^3		+	225	+	+	600	+	+	+	960	+	960	+
bc^2e		+	225	+	+	400	+	+	+	320	-	-	-
bc^2d^2		+	225	+	+	400	+	+	+	320	-	-	-

Seventh Table.

f	e	d	c	b	a	$f.1235$	$e.1236$	$d.1245$	$c.1245$	$b.1346$	$a.2356$	$b.1356$	$a.1456$
a^2cf^2	+ 10	- 390	- 90	- 114	-	- 90	+	6	+	6	+	6	+
a^2def	- 390	-	195	-	-	360	+	+ 16	-	- 24	-	- 22	-
a^2e^3	+ 155	+ 100	+ 360	-	-	195	-	6	-	4	+	16	+
ab^2cf	+ 100	- 600	- 1500	-	-	- 1500	- 22	+	+	24	+	26	+
abc^2f	- 600	+	1500	+	+	900	6	- 26	-	4	+	96	+
abd^2e	+ 1600	-	- 1500	+	+	1800	-	96	+	36	+	96	-
ac^2df	+ 1600	+	900	+	+	- 1500	+ 16	- 96	+	16	+	- 80	-
ac^2e^2	+ 1800	+	1800	+	+	1800	+	+	+	36	+	96	+
acd^2e	+	225	+	1080	+	+	+	+	+	36	+	96	+
ad^4	+ 125	- 1000	+	900	-	+	+	+	+	36	+	96	+
b^2ef	- 1000	+	225	+	+	+	+	+	+	36	+	96	+
b^2cdf		+	225	+	+	+	+	+	+	36	+	96	+
b^2ce^2		+	225	+	+	+	+	+	+	36	+	96	+
b^2d^2e		+	225	+	+	+	+	+	+	36	+	96	+
bc^2f	- 1500	-	1500	- 2700	+	- 2700	- 10	+	+	36	+	96	+
bc^2de		+	225	+	+	+	+	+	+	36	+	96	+
bc^2d^2		+	225	+	+	+	+	+	+	36	+	96	+
c^2e		+	225	+	+	+	+	+	+	36	+	96	+
c^2d^2		+	225	+	+	+	+	+	+	36	+	96	+

And the remaining seven Tables might of course be deduced from these by writing (f, e, d, c, b, a) instead of (a, b, c, d, e, f) , and making the corresponding alterations in the top line of each Table.

18. The equations $\mathfrak{A}=0, \mathfrak{B}=0, \dots, \mathfrak{M}=0$ consequently establish between the fifteen functions 1234, 1235, ... 3456 a system of fourteen equations, viz. the first and last three of these are

$$\begin{aligned}
 1234 &= 0, \\
 1235 &= 0, \\
 -160758675.1245 \\
 + 11559295.1236 &= 0, \\
 &\vdots \\
 + 11559295.1456 \\
 -160758675.2356 &= 0, \\
 2456 &= 0, \\
 3456 &= 0.
 \end{aligned}$$

To complete the proof that in virtue of the equations $\mathfrak{A}=0, \mathfrak{B}=0, \dots, \mathfrak{M}=0$ all the fifteen functions 1234, 1235, ... 3456 vanish, it is necessary to make use of the identical relations subsisting between these quantities 1234, &c.; thus we have

$$\begin{aligned}
 a.1345 + 4b.1245 + 6c.1235 + 4d.1234 &= 0, \\
 b.1345 + 4c.1245 + 6d.1235 + 4e.1234 &= 0,
 \end{aligned}$$

which, in virtue of the above equations $1234=0$ and $1235=0$, become

$$\begin{aligned}
 a.1345 + 4b.1245 &= 0, \\
 b.1345 + 4c.1245 &= 0,
 \end{aligned}$$

giving (unless indeed $ac-b^2=0$) $1245=0$, $1345=0$; the equation $1245=0$ then reduces the third of the above equations to $1236=0$, and so on until it is shown that the fifteen quantities all vanish.